Graph Theory – a survey on the occasion of the Abel prize for László Lovász

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Abstract

In this survey, we explain a few key ideas of the theory of graphs, and how these ideas have grown to form the foundation of entire research areas. Graph Theory is a fairly young mathematical discipline; here we explain some of its major challenges for the 21st century.

László Lovász was recently awarded the Abel Prize. He made important contributions to all the areas discussed in this survey, and we close by summarising his main achievements.

1 What is a graph?

A *graph* consists of a set of *vertices* together with a set of *edges* such that each edge is *incident* with exactly two vertices. These two incident vertices are called the *endvertices* of that edge. Sometimes, it will be convenient to allow these two endvertices to be the same or to allow edges with the same pair of endvertices. For now, however, we restrict our attention to graphs where this does not happen. Topologically speaking, a graph is simply a 1-dimensional simplicial complex.

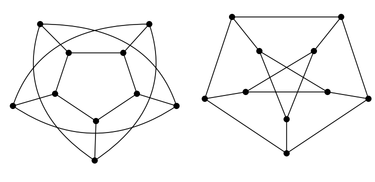


Figure 1: The graph on the left is isomorphic to the graph on the right.

Two graphs are considered the same; that is, they are *isomorphic*, if there are bijections between their vertex sets and edge sets that commute with their incidence relations, see Figure 1.

While road networks, and various networks associated to the internet give immediate examples of graphs, see Figure 2, here is an example how graphs are used in another area of mathematics. Given a presented group, the vertex set of its *Cayley graph* is the set of elements of the group, and we join two elements a and b by an edge if there is a nontrivial element s in the generating set such that $a \cdot s = b$. Many highly symmetric graphs are Cayley graphs of groups. For example the cube is a Cayley graph of the dihedral group D_4, whereas the Petersen graph depicted in Figure 1 is not the Cayley graph of any group. Of particular interest are properties of Cayley graphs that are invariant under changing the presentation, and thus they are properties of the underlying groups; for example *ends* of groups in the sense of Freudenthal [30]. The structure of ends in groups can be studied through graphs using Bass-Serre Theory[1] [84, 88].

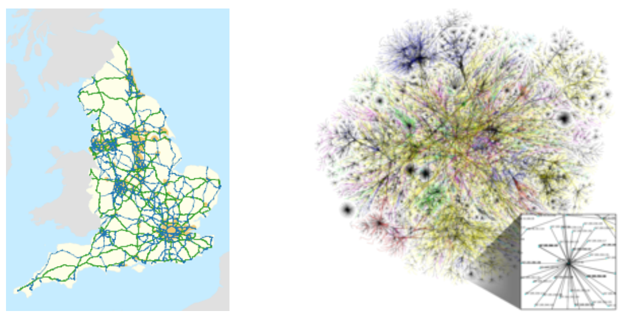


Figure 2: Graphs are used to model road networks (left) and networks associated to the internet (right), (Images from Wikipedia, the right picture is an An Opte Project visualization of routing paths through a portion of the Internet).

2 An Introduction to Graph Theory

Many questions in graph theory arise from studying the structure of natural classes of graphs. There are different approaches to deriving solutions to such questions. Over the years these approaches have been refined into a profound tool box of methods; and often graph theory is thought of as being composed of the subfields that emerged out of these approaches. In what follows I will introduce some of the

[1] *Bass-Serre theory* is a subfield of combinatorial group theory that deals with analysing groups acting by automorphisms on trees. Tools in this area allow us to detect free products with amalgamation or HHN-extensions.

basic questions of graph theory, and explain how its subfields arose by providing answers to these questions.

Extremal Graph Theory. A natural ordering on graphs is the 'subgraph relation'; here we say that a graph H is a *subgraph* of a graph G if H can be obtained from G by deleting edges and deleting isolated vertices – that is, vertices not incident with any edges. Given a natural number r, the class of graphs with r vertices has a unique maximal graph with respect to the subgraph relation (up to isomorphism). This graph has edges between all $\binom{r}{2}$ pairs of vertices; it is referred to as the *complete graph* on r vertices, denoted by K_r. It is natural to ask: which graphs contain the complete graph K_r as a subgraph? Or restating it through the complementary class: what is the structure of the class of graphs that do not have K_r as a subgraph? While this class of graphs seems to be pretty wild and a complete characterisation of the class may well be elusive, in 1941 Turán provided the following partial answer (extending an earlier observation of Mantel from 1907).

Consider a graph G formed by $r-1$ classes of vertices such that two vertices are adjacent if and only if they are in different classes (Here two vertices are *adjacent* if they are joined by an edge), see Figure 3. Clearly the graph G does not have the complete graph K_r as a subgraph. We say that G is a *Turán-graph* if any two of its $r-1$ classes of vertices differ in size by at most one. Note that up to isomorphism, there is only one Turán-graph on a fixed number $n > r$ of vertices. In 1941 Turán proved that any graph on $n > r$ vertices that does not have the complete graph K_r as a subgraph and has *as many edges as possible* must be isomorphic to the Turán-graph on n vertices.

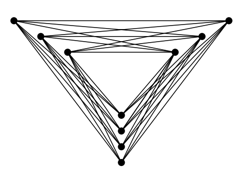


Figure 3: The Turán graph for $r = 4$ on 10 vertices.

Turán's theorem is the foundational example for the approach of *Extremal Graph Theory.* A key tool in extremal graph theory is Szemerédi's *regularity lemma.* This says that the vertex set of every large-enough graph G can be partitioned into few subsets so that the edges of G between almost all of these subsets are distributed 'regularly': as one would expect it if they were picked at random, given the actual density of the edges of G between those two sets of vertices. Due to this inherent randomness, results in extremal graph theory that rely on the regularity lemma in their proof are often only asymptotic and approximate; that is, they are only valid for graphs on sufficiently many vertices and quantify by how many edges those graphs differ from a list of constructions.

To summarise, in extremal graph theory we aim to draw structural conclusions (such as the existence of a K_r subgraph) from merely quantitative assumptions (such as having more edges on n vertices than the Turán-graph does). Many of its methods are closely related to probability theory.

Structural Graph Theory. In the early days of graph theory, an influential problem was the 4-Colour Conjecture. Informally speaking, it says that the countries on any map can be coloured with at most four colours so that adjacent countries receive different colours, see Figure 4.

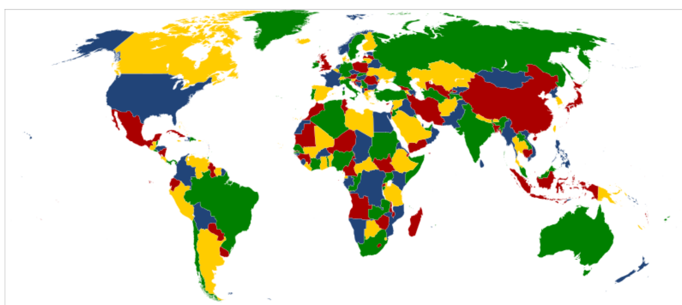


Figure 4: A colouring of the countries of the world with just four different colours (Source: Wikipedia).

This problem can be modelled by a graph. Just assign a vertex to each country and join two of these vertices by an edge if their corresponding countries share a border. This graph has the property that it is *planar*; that is, its geometric realisation (the associated 1-dimensional simplicial complex) can be injectively and continuously embedded in the two-dimensional plane, see Figure 5. Hence the formal statement of the 4-Colour Theorem is that the vertex set of any planar graph can be partitioned into four classes so that no edge has both its endvertices in the same class. In 1976, well over a hundred years after it had been proposed, the 4-Colour Theorem was proved by Appel and Haken. Next to Haken's foundational contributions to computational topology, in particular the theory of normal surfaces, the 4-Colour Theorem is regarded as one of his main achievements. As for Hales' proof of Kepler's Conjecture, so far we only know computer-assisted proofs of the 4-Colour Theorem, which may be unexpected given the statement of the 4-Colour Theorem.

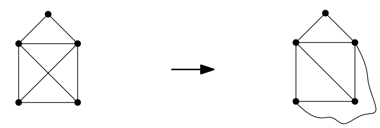


Figure 5: A planar graph with an embedding.

Possibly motivated by the 4-Colour Conjecture, in the 1930s mathematicians like Kuratowski, Wagner and Whitney initiated the systematic study of the class of planar graphs. As we shall see, these investigations eventually led to the development of Graph Minor Theory, which nowadays is considered as far more important than the 4-Colour Theorem itself, the intriguing puzzle with which all these developments started.

In the above definition of 'planar graphs' we might as well have defined them slightly differently, requiring that the embedding of the edges is not only continuous but differentiable, or piece-wise linear, or even that the edges are embedded as straight lines. In 1943 Koebe proved that all these notions of embeddability are equivalent (for finite graphs, for infinite graphs see [42]) by proving that they are equivalent to an even stronger notion of embeddability. After being forgotten due to the horrible catastrophy of World War Two, this theorem was rediscovered independently by Andreev and Thurston and subsequently it was shown by Rodin and Sullivan in 1987 that this theorem can be applied in differential geometry to obtain a short combinatorial proof of the Riemannian Mapping Theorem [78], see also [43]. To summarise, there are many potential definitions of planar graphs, and it has been shown that they are all equivalent (for finite graphs), leading to a single class of planar graphs.

Making use of Euler's Polyhedra Formula, it is an easy exercise to show that the complete graph K_5 is not planar. Similarly, the complete bipartite graph $K_{3,3}$, see Figure 6, is not planar.

Since subgraphs of planar graphs are planar, it may be tempting to try to characterise the class of planar graphs by making a list of all the minimally non-planar graphs: those non-planar graphs all whose proper subgraphs are planar. Unfortunately, the list of these graphs is far too complicated. From a structural perspective, the reason for this is that the subgraph relation is not closed under planar duality

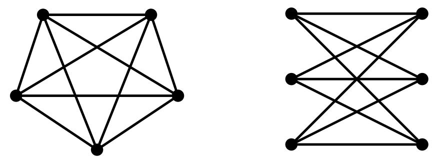


Figure 6: The graphs K_5 (on the left) and $K_{3,3}$ (on the right).

in the following sense: start with a graph G embedded in the plane, take its planar dual, delete an edge, and take the planar dual again. The resulting graph is always planar but need not be a subgraph of the graph G we started with, see Figure 7.

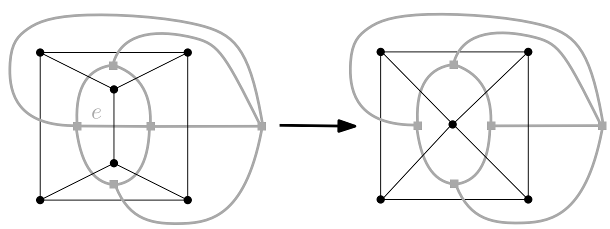


Figure 7: The graph G is the black graph on the left. Its plane dual is depicted in grey in the same figure. The grey graph on the right is obtained from the grey graph on the left by deleting the edge e. The black graph on the right is dual of the grey graph on the right. However, it is not a subgraph of the graph G we started with.

In 1933, Wagner introduced the 'minor relation' as a refinement of the subgraph relation that solved the above problem resulting from the deficit of the subgraph relation that it does not 'behave well with planar duality', as follows. A *minor* of a graph is obtained by deleting edges and contracting connected[2] edge sets to single vertices, see Figure 8. Minors of graphs may have multiple edges between a pair of vertices and may have edges whose two endvertices are the same; this slight extension of the class of graphs is referred to as the class of *multigraphs*[3]. The difference between graphs and multigraphs is mostly of a technical nature and with slight abuse of notation we shall not distinguish between the two in this survey; we shall always use the term 'graph' although sometimes the objects can actually be multigraphs.

[2] An edge set is *connected* if any two of their endvertices can be joined by a path.

[3] In Graph Minor Theory, the term 'graph' is used for 'multigraphs', while 'graphs' as we defined them here are referred to as 'simple graphs'.

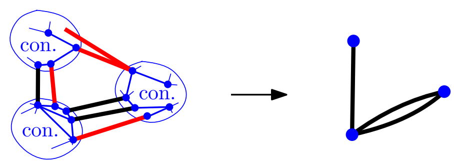


Figure 8: The minor of the graph on the left obtained by deleting the red edges and contracting the blue edges is depicted on the right.

For planar graphs, the minor relation is obtained from the subgraph relation by closing it under planar duality (Figure 9), but note that the minors are defined combinatorially for arbitrary graphs.

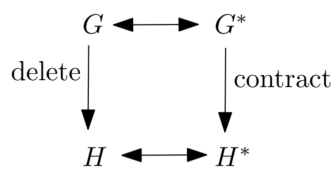


Figure 9: Contracting an edge in a planar graph is the same operation as dualising, deleting that edge in the dual, and then dualising back; in other words the diagram in this figures commutes. See Figure 7 for an example for contraction of an edge.

Planarity of graphs can be characterised through the minor relation: a graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor. With a slightly different notation, this was first proved by Kuratowski [51], and it is referred to as 'Kuratowski's Theorem'. This theorem is an example of what Edmonds calls a *good characterisation*[4]. Given a graph, if it is planar, then there is a simple certificate for it: just give the drawing. If it is not planar, by Kuratowski's theorem, there is also a simple certificate for it: it must have one of the two graphs K_5 or $K_{3,3}$ as a minor.

In addition to the class of planar graphs, there are quite a few other natural classes of graphs that are closed under the minor relation, for example the class of graphs embeddable in any fixed surface. *Graph Minor Theory*, initiated through the works of Kuratowski and Wagner in the 1930s, investigates the minor relation on general graphs.

A far-reaching generalisation of Kuratowski's Theorem is the Robertson-Seymour Theorem. This theorem is concerned with general classes of

[4]Formally, a problem has a *good characterisation* if it is in the complexity class NP $\cap$ co-NP [23, 55, 96].

graphs closed under taking minors. Each such class is characterised by its *excluded minors*, the minor-minimal graphs outside it. This theorem says that for any minor-closed class its list of excluded minors must be *finite*. While the proof easily spans over 500 pages, at the core of the proof of this theorem is the *Graph Minor Structure Theorem*, which in a sense gives a topological construction rule for any minor-closed class, establishing a deep connection between graph theory and topology.

To summarise, in Structural Graph Theory we aim to relate structural graph properties to one another, such as embeddability in a surface or the existence of certain minors. Often, results in this area are related to other areas of mathematics, such as topology, geometry or algebra, or to the design of efficient algorithms.

What is the difference between Extremal Graph Theory and Structural Graph Theory? There are many results in the intersection of these areas. In Extremal Graph Theory, theorems typically relate a numerical graph invariant, such as the number of edges, to a structural one, or even take the form of an inequality between two numerical graph invariants. Results are relatively easy to compare through these estimates. Typical methods, such as Szemerédi's regularity lemma, are sometimes not even referred to as 'theorems', and often applying these methods requires a fair amount of computation and estimating.

In Structural Graph Theory, theorems typically establish a connection between two structural graph properties; such results are then used directly to prove further results, forming a diverse landscape of interconnected theorems.

Both approaches have their advantages, and it depends on the type of problem you are trying to solve which is more suitable.

3 Highlights in Extremal Graph Theory

Ramsey Theory. Today, with data science emerging rapidly as a new discipline of science, it is time to refine our mathematical understanding of what types of structure occur *necessarily* in any large-enough sample.

In 1930, Ramsey proved that for every natural number r there is a number n – much larger than r – such that *every* graph with at least n vertices either contains the complete graph K_r or its complement, which consists of r vertices with no edges in between [72].

Ramsey's theorem is the first of its kind, in fact it finds certain pre-determined substructures in all large-enough graphs. In addition to numerous applications within combinatorics, extensions of Ramsey's theorem to infinite sets provide tools in set theory, and there is a close connection between Ramsey's theorem and Van der Waerden numbers for arithmetic progressions in number theory. Natural variants of Ramsey's Theorem suggest themselves and have been studied a lot in the past century, and are usually subsumed to form the field of 'Ramsey Theory'.

While Ramsey's theorem as such is not too difficult to prove, the quantitative behaviour of this phenomenon seems to be particularly difficult to handle; more about this in a moment. Given r, the *Ramsey number* is the smallest number n such that every graph on n vertices contains a K_r or its complement, see Figure 10. The Ramsey number is denoted by $R(r)$. The observation that Ramsey numbers, even for pretty small values of r, are difficult to compute was popularised by Paul Erdős [87].

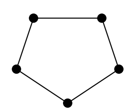


Figure 10: The 5-cycle is isomorphic to its complement and does not contain a complete graph K_3. Hence the Ramsey number $R(3)$ is at least 6; and it is indeed an easy exercise to check that the complete graph K_3 appears in any graph on six vertices or its complement as a subgraph.

While determining particular Ramsey numbers for $r \geq 6$ exactly is certainly not an easy task, the real challenge here is to understand their asymptotic behaviour. The known proofs give an exponential upper bound of the order 2^{2r}. The best known lower bounds are based on randomly constructed graphs, and are of the order of magnitude $2^{r/2}$.

Open Question 3.1. *Can you find accurate asymptotic upper and lower bounds for the Ramsey numbers $R(r)$?*

More precisely[5], *determine a constant c such that $R(r) = \Theta(2^{c \cdot r})$.*

By the above, we know that $\frac{1}{2} \leq c \leq 2$. Combining results of Spencer [87] and Conlon [20] gives the best known bounds to date:

$$[1 + o(1)] \frac{\sqrt{2}r}{e} 2^{\frac{r}{2}} \leq R(r) \leq r^{-(c \log r)/(\log \log r)} 4^r$$

It seems that in order to improve the bounds for c fundamentally new ideas are required. Interestingly, as soon as we assume some struc-

[5]Given functions f and g, we write $g = \theta(f)$ as a shorthand for: there are constants a and b such that $a \cdot f \leq g \leq b \cdot f$.

ture for the graphs in which we seek to determine c, it is possible to prove upper bounds that are much lower.

For example, in the simplified Ramsey problem for triangle-free graphs much better quantitative bounds are known; see recent work of Bohman and Keevash [8] and Ajtai, Komlós and Szemerédi [5].

A useful strengthening of the notion of a subgraph is that of an *induced subgraph*: a graph obtained from another graph by deleting some of its vertices, and all their incident edges. Given a graph H, we denote by $\mathcal{F}_H$ the class of graphs with no induced subgraph isomorphic to H. While random graphs have fairly large Ramsey numbers, as mentioned above, Erdős and Hajnal conjectured that in any class $\mathcal{F}_H$ – which cannot contain large random graphs, in the sense that those contain every fixed H with probability tending to 1 as their size grows – Ramsey numbers restricted to that class are reasonably small:

Conjecture 3.2 (Erdős-Hajnal Conjecture 1989)**.** *For all graphs H, there exists a constant $\delta_H > 0$ such that the n-vertex graphs in $\mathcal{F}_H$ contain either a complete graph, or its complement, of size $\Omega(n^{\delta_H})$ as an induced subgraph.*

While there are a few theorems proving this conjecture for particular graphs H, this conjecture remains widely open; see Chudnowski's survey [15] for details.

Probabilistic Method. There are many beautiful conjectures out there that only have one little problem: they are not true. And we all know that sometimes it can be hard to find a construction for a counterexample. The 'Probabilistic Method' provides a systematic way to produce counterexamples: rather than constructing a concrete counterexample explicitly, one proves that they occur with some positive probability and hence must exist. Over the years this method has been applied successfully to a large class of problems in combinatorics.

Let us start at the beginning, with a conjecture that is just too beautiful to be true. A *colouring* of a graph is an assignment of colours to its vertices such that adjacent vertices receive different colours. The *chromatic number* of a graph is the least number of colours required. For example, the complete graph K_r has chromatic number r, while trees – that is, connected graphs without cycles – have chromatic number two (to see this, pick a vertex of the tree, call it the *root*, and assign one colour to all vertices of even distance from the root and the other colour to the vertices of odd distance). Above we discussed the 4-Colour Theorem, which says that planar graphs have chromatic number at most four.

Assume our task is to colour a huge graph with a small number of colours and we are given colourings with just two colours of all connected subgraphs of some bounded size. One might imagine that there would be a 'local-global principle' allowing us to stick together the local colourings of the bounded-sized graphs to produce a global colouring with two colours – or perhaps, allowing for a small slack, with boundedly many colours, see Figure 11. In 1959 Erdős showed that such a strategy cannot work, by proving probabilistically the existence of a large class of counterexamples. He showed that for any number k, there are graphs of chromatic number k such that all connected subgraphs on at most k vertices are trees. Put another way, local 2-colourings of graphs cannot always be combined to global colourings with just boundedly many colours.

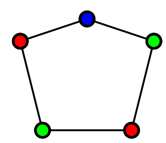


Figure 11: Although every path can be coloured by alternating between two colours, cycles of odd lengths cannot be coloured using only two colours. They are the simplest examples of graphs where local reasons alone do not determine the chromatic number.

How do we randomly generate such a graph? In the simplest (Erdős-Renyi) model, we start with a set of n vertices and put in each possible edge with the same probability p – which may depend on n. With some abuse of terminology, we then say that a *random graph* has a particular property if the measure of the set of n-vertex graphs with that property in the resulting probability space $\mathcal{G}(n,p)$ tends to 1 as n tends to infinity, see Figure 12.

We can thus estimate probabilities for events like 'containing no cycle of length at most k', or of 'having chromatic number at least k'. If both these probabilities were greater than one-half, then the existence of a graph with both properties is proved: the existence of a graph that is locally 2-colourable (because it contains no 'short' cycles) yet needs many ($> k$) colours globally. Unfortunately, in reality things are slightly more complicated; see [23] for the easy but beautiful details.

In our examples the probabilistic method was introduced to construct graphs with two properties simultaneously. In 1973, more than twenty years after the probabilistic method had been introduced to graph theory, Erdős and Lovász initiated a more systematic study of probabilistic constructions in combinatorics, as follows.

The Chernoff bound from Probability Theory is a commonly used tool to control the joint distribution of many random variables – as long as they are independent. However, in many potential applica-

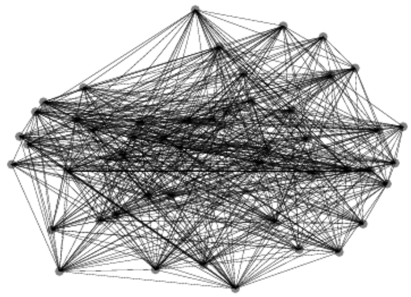


Figure 12: A graph on $n = 50$ vertices generated in such a way that every edge was added with probability $p = \frac{1}{2}$ (Source: [82]). It is unlikely that two graphs generated independently in this way are isomorphic. In contrast to this, any two random graphs on a countably infinite vertex set are isomorphic with probability one [1, 29, 71].

tions in graph theory, we would like to control lots of events but they are not all independent. Still most of the pairs of these events are independent. In this situation, Erdős and Lovász decided to encode this information about the dependencies in a graph, and study which structural property of this dependency graph allow for a solution to all the constraints given by the events. More precisely, the Lovász Local Lemma says that given a family of events such that their dependency graph has maximum degree d and all these events occur with probability at most p such that $ep(d + 1) \leq 1$ (where $e = 2.7182..$), there is a positive probability that none of these events occurs.

For example, initially with the probabilistic method one was able to prove only the non-existence of a colouring with few colours, in contrast to this the Lovász Local Lemma can be used to prove the *existence* of such a colouring, significantly expanding the potential applications. In a nutshell, the probabilistic method grew from a useful tool to construct counterexamples to persistent conjectures, through Lovász' contributions, to one of the key construction methods in the area of Probabilistic Algorithms. While new applications of the Lovász Local Lemma are still being discovered today, Moser and Tardos found a constructive version of the Lovász Local Lemma in 2010 [62, 63].

Limits of graphs. Given that many theorems in graph theory are asymptotic in nature, it is a natural step to study these problems through a suitable limit object. Asymptotic problems for dense graph classes – classes consisting of graphs whose number of edges is a constant fraction of all possible edges – are quite often studied through *graphons*, which are symmetric measurable functions from the unit square to the unit interval. Indeed, finite graphs can be represented by the following graphons W: partition the unit interval into equally sized segments I_v, one for each vertex v. Now define the graphon $W(x, y)$ to be one if there is an edge between the vertices v and w satisfying $x \in I_v$ and $y \in I_w$, and zero otherwise. Another example of a graphon is given in Figure 13.

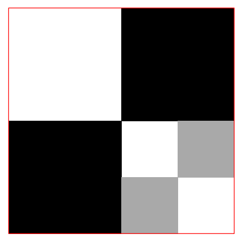


Figure 13: A graphon is a function from the unit square to the unit interval. We draw them as a unit square, whose points are coloured black if they attain the value 1, white if they attain the value zero, and various shades of grey for all values in between. The graphon here represents the class of graphs that can be partitioned into two equally sized vertex sets such that all edges between them are present. In one of these vertex sets no edge is present. The other vertex set can be partitioned into two independent sets such that an edge between these sets is present with probability one half.

Given a sequence of finite graphs, represented by graphons taking only the values zero and one, there is a suitable notion of a limit graphon, similar as the example of Figure 13, whose values represent densities from the unit interval obtained by averaging out vertex-adjacencies, roughly speaking; see the foundational book of Lovász [56] for details.

This way a whole class of graphs can be represented by a single graphon. For example, the graphon that takes constantly the value $\frac{1}{2}$ represents the class of Erdős-Renyi random graphs, where edges are drawn with probability $\frac{1}{2}$. Graph parameters such as edge densities or numbers of triangles can be interpreted as integrals over graphons. This allows us to translate many problems from extremal graph theory into inequalities over multidimensional real-valued integrals.

This topological construction of graphons as limits is accompanied by algebraic constructions, see the flag algebra calculus of Razborov [73]. In some cases the automatisation of extremal problems through limits is at a level that parts of the problem solving can be done through computer search [74]. The theory of graphons is closely related to Szemerédi's regularity lemma. For example, in this language the regularity lemma can be restated in the form that every arbitrary large graph can be approximated (in a suitably quantified way) by a graphon given by a partition of the unit square into boundedly many equally sized squares that is constant on all these partition classes.

Within this framework of graph limits, in 2016 Reiher proved his clique-density theorem, a far-reaching extension of Turán's classical theorem, which for every natural number r determines the minimal number of complete graphs on r vertices that are contained in any graph with a given edge density [75]. This solved a conjecture of Lovász and Simonovits from the 1970s and extended earlier results of Razborov and Nikiforov. An important conjecture in this area that remains is the following. Given a graph H and a graph G, denote by $t(H, G)$, the 'homomorphism density of H in G' – up to some low-order terms this is equal to the probability that if we pick a set of $|V(H)|$ vertices of G at random that the subgraph of G spanned by these vertices has the graph H as a subgraph.

Conjecture 3.3 (Sidorenko 1986 [86])**.** *For any bipartite graph H and every graph G we have that:*

$$t(H, G) \geq t(K_2, G)^{|E(H)|}$$

Informally, this conjecture says that in every graph G, the edge-density $t(K_2, G)$ can be used lower-bound densities of arbitrary bipartite graphs H in the most natural way. In additional to his foundational contributions to the development of the theory of graphons, Lovász asked questions about them with the aim to determine the potential and boundaries of this approach, quite a few of them have been answered recently by Král' et al [22, 39].

The theory of dense graphs can be used to solve problems in all types of areas, for example it can be applied in number theory to study arithmetic progressions in random subsets of the integers; see for example the works of Schacht [80] and Conlon and Gowers [21].

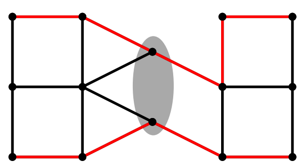
While the methods of graphons and flag algebras are tailored to dense classes of graphs, a theory of limits for sparser classes of graphs has not been fully developed yet [50].

Open Question 3.4. *Can you develop a theory of limits of sparse graphs that provides a systematic framework to study a great variety of asymptotic questions on sparse graphs?*

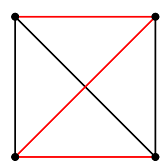
One hope of such a theory of sparse graphs would be that it might provide tools to study Gromov's question whether all finitely presented groups are sofic [37] through the Aldous-Lyons Conjecture [6], see [41]. For the emerging theory of sparse graphs see the book of Nešetřil and Ossana de Mendez [68]. The subfield of Structural Graph Theory that traditionally deals with problems of limits of graphs is called *Infinite Graph Theory*, which we explore in a separate paragraph below.

4 Highlights in Structural Graph Theory

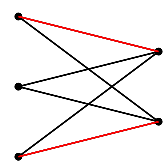
Matchings, Packings and Coverings in Graphs. Graph theory is a fairly young mathematical discipline, many of whose foundational results were proved in the 1930s and 1940s. Now it has reached a stage with an abundance of applications in science and connections to other areas of mathematics, where its theorems accumulate and greater patterns between the theorems are recognised. Let me try to illustrate this by explaining four theorems answering four seemingly very different problems, and their common unification.



(a) There cannot be more than two paths from left to right as all such paths need to traverse the 2-separator, highlighted in grey.



(b) The edge set of the graph K_4 can be covered by two paths of length three; this covering happens to be also a packing.



(c) The complete bipartite graph $K_{2,3}$ has a matching of size 2; one such is highlighted in red.

Figure 14: Examples related to (a) Menger's Theorem, (b) the Covering Theorem and (c) the Marriage Theorem, all special cases of the Packing Covering Theorem.

Firstly, given a graph G together with two vertex sets A and B, what is the largest size of a set of vertex-disjoint paths from A to B? In 1927 Menger answered this question by proving that the largest cardinality of a set of vertex-disjoint paths from A to B is equal to the minimum size of a vertex set whose removal separates A from B [59]; note that it clearly cannot be larger, see Figure 14a.

Secondly, given a graph G, how many subtrees of G are necessary to cover all its edges? Note that when covering the edges of a (connected) graph with trees, we may as well assume that each of these trees contains all vertices of the graph; that is, it is a *spanning tree*. In 1964 Nash-Williams [65] answered that question by proving that the minimum number of trees necessary is no larger than the maximum local density of the graph – the largest ratio of the edges over vertices in any subgraph and which clearly is a lower bound since in trees this number is less than 1, see Figure 14b.

Thirdly, given a connected graph, what is the maximum number of pairwise edge-disjoint spanning trees? Nash-Williams and Tutte [64, 91] answered this question by proving that this number is equal to the minimum local density of a contraction minor[6] (that is, a minor obtained by only contracting edges).

Fourth, in a small village, what is the maximum number of marriages that can be arranged between men and women, given a graph encoding the possible matchings? Formally, we are given a *bipartite graph*, a graph whose vertex set is partitioned into two classes such that all edges go between these classes; and we are interested in finding a set of vertex-disjoint[7] edges – such a set of edges is called a *matching*. In 1931 Kőnig answered this question by proving that the maximum size of a matching is equal to the minimum size of a vertex set covering all edges, see Figure 14c. Independently in 1935, Hall proved an equivalent result, which has been popularised as the 'Marriage Theorem'. Most certainly it was clear at that time that matchings have numerous applications outside mathematics, for example in scheduling problems. An application within mathematics is a short combinatorial proof of the Cantor-Schröder-Bernstein Theorem in set theory. For details on the theory of matchings, we refer to the book of Lovász and Plummer [57].

Despite their differences, all these four theorems have the following unification [9, 23]. Given a graph G and a natural number k, its edge set can be partitioned into a set P and a set C satisfying the following:

- the graph $G\restriction P$ – the subgraph of G obtained by deleting the edges outside P – has k edge-disjoint spanning trees in each of its connected components. The edge set P is referred to as a *packing*;
- the graph $G_i.C$ – the minor obtained from G by contracting the edges outside C – has a covering of its edge set by k trees. The edge set C is referred to as a *covering*.

Informally speaking, this theorem says that the edge set of any graph can be partitioned into a *dense* part, which admits a packing, and a *sparse* part that admits a covering. The *Packing-Covering Theorem* is the natural generalisation of this statement where one takes k matroids sharing a common ground set instead of a single graph G. While the reductions of the above four theorems from the Packing-Covering Theorem are automatic once we specify to which family of graphs (or matroids) we apply this theorem, the choice of a suitable family requires a little bit of thought.

[6]There is a variant of this theorem with 'contraction minor' replaced by 'quotient' [23]. These two versions are clearly equivalent.

[7]We say that a set of edges is *vertex-disjoint* if no two edges in that set share a vertex.

Quite often in graph theory, people only consider finite graphs – it is the convention to always assume that graphs are finite unless stated explicitly that they are infinite. In the Packing-Covering Theorem allowing the set E to be infinite, one obtains the *Packing-Covering Conjecture* [9], which is still open; important special cases of this conjecture include the Aharoni-Berger Theorem from 2009 [4], which is an infinite analogue of Menger's Theorem conjectured by Erdős in the 1960s, and it is a far-reaching extension of the Infinite Hall Theorem of Aharoni, Nash-Williams and Shelah from the 1980s. Quite recently Joó [45] made a big advance on the Packing-Covering Conjecture.

In its most general form, the Packing Covering Theorem is equivalent to the Matroid Intersection Theorem. This theorem expresses these ideas in terms of submodular rank functions; here a function f from the power set 2^E of a set E to the reals is *submodular* if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq E$. The *Lovász extension* of a function f from 2^E to the reals is the function $\widehat{f}$ from the unit cube $[0,1]^E$ to the reals given by $\widehat{f}(x) = \mathbb{E}[f(x_\lambda)]$, where the expectation is taken over the parameter $\lambda \in [0,1]$ and x_λ is the binary vector that has a one in all coordinates where x has a value of at least λ [54]. This construction is designed so that the function f is submodular if and only if its extension $\widehat{f}$ is convex. This established an important link with optimisation, allowing us to derive integral solutions for a large class of graph-theoretic problems through fractional relaxations thereof [38]. Another connection with the related field of *Combinatorial Optimisation* is described in the paragraph on perfect graphs below.

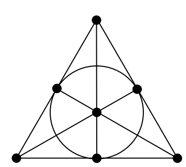


Figure 15: The fano plane is a Steiner-system with parameters $(7, 3, 2)$; here a *Steiner-system* with parameters (n, q, r) is a set S of q-subsets of an n-set X, such that every r-subset of X belongs to exactly one element of S. Steiner-systems have applications in error-correcting codes.

While the Packing-Covering Theorem gives us a good understanding of how to pack spanning trees in graphs, it is natural to try to pack other graphs. In *Design Theory*, we are given a large host graph G and a small graph F. The task is to partition the edge set of G into graphs all isomorphic to F. For simplicity we assume here that all the vertices of G and F have the same number of neighbours, and denote these numbers by $d(G)$ and $d(F)$, respectively. For such a partition of the edge set G to exist, it is necessary that $d(F)$ divides $d(G)$, and that the number of edges of F divides the number of edges of G. In this case we say that G is *F-divisible*. In 1853 Steiner conjectured that if G and F are both complete graphs such that G is F-divisible and G and F are sufficiently large, then the edge set of G can be partitioned into copies of F. In 2014 Keevash [47] announced a proof of this conjecture (in a more general form including Steiner triples, see Figure 15 for details), which is widely believed to be correct though still under review. For general graphs this problem is widely open:

Conjecture 4.1 (Nash-Williams 1970 [66], generalised by Gustavsson [40] beyond the case r=3)**.** *For every $r \geq 3$, there exists a constant $n_0 = n_0(r)$ such that every K_r-divisible graph G with $n \geq n_0(r)$ vertices such that every vertex has at least $(1 - \frac{1}{r+1}) \cdot n$ neighbours admits a partition of its edge set into graphs all isomorphic to K_r.*

See [36] by Glock, Kühn, Lo, Montgomery and Osthus for recent progress towards this conjecture.

Graph Minor Theory. Graph Minor Theory sits at the interface of topology and graph theory, with many algorithmic applications. An important aspect is the connection between embeddings of graphs in 2-dimensional surfaces and the minor relation. The discovery of this connection began in the 1930s when the pioneers Wagner, Kuratowski and Whitney characterised embeddability of graphs in the plane by completely combinatorial conditions [94, 51, 95] and this connection provided crucial tools for the proof of the Robertson–Seymour Theorem [77] in 2004, which is often regarded as the deepest theorem of combinatorics today. On one hand, the class of graphs embeddable in a fixed surface is closed under taking minors. This means that, like for Kuratowski's Theorem, embeddability in a fixed surface can be characterised through the minor relation in terms of a list of minimal graphs that do not embed; such graphs are called *excluded minors*. The Robertson-Seymour theorem says that this list of excluded minors is *finite* for every minor-closed class of graphs. On the other hand, Robertson's and Seymour's structure theorem provides a topological construction for every minor-closed class of graphs; drastically oversimplifying, this theorem says that any minor-closed class of graphs can be built from classes of graphs embedded in a fixed surface by sticking them together in a tree-like way, see Figure 16.

This theoretical breakthrough demonstrates the potential of the minor-theoretic approach. Still it seems like we are very much at the beginning of constructing *Minor Theory*, as the current machinery cannot be used to derive quite a few seemingly natural applications. For example, while Mohar proved that there is a linear time algorithm that

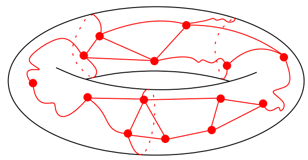


Figure 16: A graph embedded in the torus. This embedding induces a toridal embedding of every minor of this graph.

verifies embeddability in a fixed surface [61, 46] and the Robertson-Seymour theorem predicts that the class of graphs embeddable in a fixed surface is characterised by a finite list of excluded minors, computing this list explicitly for a fixed surface is a challenge – even for the simple surface of the torus this list is not known! Another example is Negami's Conjecture on coverings of planar graphs from 1988 [67]. For Hadwiger's Conjecture see the paragraph on 'Graph Colouring'.

The spectral theorist Colin de Verdière, introduced the parameter $\mu(G)$. It originated from the study of the maximum multiplicity of the second eigenvalue of certain Schrödinger operators on Riemann surfaces [93]. A discretisation led to a graph-theoretic description of this quantity. Very roughly, $\mu(G)$ is the largest co-rank of a class of generalised adjacency matrices satisfying a certain linear algebra property called the 'Strong Arnol'd Property'. While this definition is purely given in terms of linear algebra, it can be shown that the class of graphs with $\mu(G) \leq c$ for some constant c is closed under taking minors. So by the Robertson-Seymour Theorem the condition that '$\mu(G) \leq c$' can be characterised by finitely many excluded minors and the structure theorem suggests that this class can be characterised in topological terms. For small values of c it has been shown that this is indeed the case, for example for $c = 3$ the class of graphs G with $\mu(G) \leq c$ is equal to the class of planar graphs, while for $c = 4$ this class is the class of linklessly embeddable graphs. For $c = 5$ so far there is only a conjecture. We say that a graph G is *4-flat* if the 2-complex obtained from G by gluing a disc onto each of its cycles is embeddable in 4-dimensional space (in a piece-wise linear way).

Conjecture 4.2 (Van der Holst 2006 [92]). *A graph is* 4*-flat if and only if* $\mu(G) \leq 5$.

There are quite a few areas of Graph Minor Theory, where exciting new structural methods are being developed. For example, the dichotomy between tree-structure and highly connected clusters led to the Tangle-Tree Theorem, the Cops-and-Robbers Theorem, the Branch Width Theorem, and the Grid Theorem. The Grid Theorem [26] roughly says that for every number n there is a number t such that every graph can be constructed from graphs of size at most t by gluing them together along a tree or it has a 2-dimensional $n \times n$ grid as a minor. Of particular interest in this dichotomy result between tree-structure and grid minors is the dependence between the parameters n and t; recently it was proved by Chuzhoy et al [14, 19] that this dependence is polynomial.

A central aim in network theory is to identify 'clusters'; that is, somewhat highly connected regions in networks. While clusters themselves are usually 'fuzzy' in the sense that it is hard to decide where they have their boundary exactly, this does not mean that they must have a fuzzy definition. One type of such clusters are *tangles*, which obey a precise mathematical axiomatisation and are key in Robertson's and Seymour's proof of their structure theorem to detect large grid minors. Diestel et al developed a systematic theory of tangles [24] that can also be applied in areas outside Structural Graph Theory such as image recognition [28, 25, 27].

Finally, there are ideas – at various stages – to extend the theory beyond graphs. These include extensions to other natural relations on graphs such as vertex minors and pivot minors by Oum et al [69, 34, 48], extensions to directed graphs by Archontia, Kawarabayashi, Kreutzer and Kwon [35], extensions to 2-dimensional simplicial complexes (see the paragraph on 'Topological Graph Theory' for details) and extensions to matroids, as follows.

Like the abstraction that came with a base point free axiomatisation of vector spaces deepened our understanding of linear algebra, *matroids* can be thought of as a 'vertex-free' abstraction of graphs. Matroids are fairly general objects which provide a unified way to understand cycles in graphs and linear dependences in vector spaces, which also captures more complicated algebraic constructions like those coming from field extensions [70]. An exciting conjecture in this area is Rota's basis conjecture (not to be confused with Rota's Well-quasi-ordering conjecture mentioned below); for an approach through extremal combinatorics, see [11] authored by Bucić, Kwan, Pokrovskiy and Sudakov.

Geelen, Gerards and Whittle announced their proof of Rota's Well-quasi-ordering conjecture [31], a far-reaching extension of the Robertson-Seymour Theorem from graphs to matroids representable over finite fields – which can be thought of as a well-quasi ordering result for matrices over a fixed finite field, roughly speaking. Matroid Minor Theory is emerging as a research area of its own, which can be applied in coding theory as well as extremal matroid theory, see for example the growth rate theorem for matroids by Geelen and Nelson [33, 32].

Graph Colouring. In Graph Colouring we study the chromatic number and related graph parameters, for example fractional relaxations thereof and inductive strengthenings like list-colourings as well as flows, which can be understood for plane graphs through the chromatic number of their plane dual, see Figure 17. While the proof of the 4-Colour Theorem in the 1970s resolved a long standing conjecture, during these investigations lots of related questions were asked, some of which go far beyond the 4-Colour Theorem and motivate a lot of research in the field of Graph Colouring till today. Examples include Thomassen's theorem that planar graphs are 5-list colourable [89] or Tutte's flow conjectures [23].

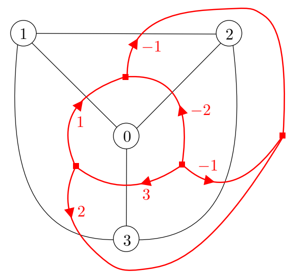


Figure 17: The vertices of the black graph K_4 are assigned colours from the set $\{0, 1, 2, 3\}$. The dual graph, which is also a K_4, is drawn in red. The colouring of the black K_4 induces a *flow* on the red K_4, as follows. Each red directed edge crosses a unique black edge. It is assigned the value of the left endvertex of that edge minus the value of the right endvertex of that edge, viewed from the crossing point in the direction of the red edge. This is a flow on the red graph as the amount flowing into each red vertex is equal to the amount flowing out of it. Tutte's duality theorem formalises a duality between colourings of plane graphs and flows in their duals. Through this duality, the 4-Colour Theorem can be restated as a theorem about the existence of certain flows in plane graphs. Tutte made various related conjectures about flows in graphs.

Amongst such questions, Hadwiger's conjecture – which seeks for a structural understanding of the chromatic number – has probably attracted the most attention. This development began in 1937 when Wagner proved that the 4-Colour Conjecture is equivalent to the statement a graph with no K_5-minor is 4-colourable [94], de-topologising the 4-Colour Conjecture. Inspired by this, Hadwiger made the following bold conjecture:

Conjecture 4.3. *(Hadwiger 1943) For all $t \geq 0$, every graph of chromatic number t has a complete graph K_t as a minor.*

In 1993, Robertson, Seymour and Thomas proved the case $t = 5$

[76]. While this conjecture certainly belongs in the area of Structural Graph Theory, a hot topic are quantitative relaxations of this conjecture; so proving statements of the form, chromatic number $f(t)$ implies a K_t-minor with the goal to eventually prove such a result for the function $f(t) = t$. See Seymour's survey [85] for details.

Perfect Graphs. Quite a few important theorems in combinatorics can be stated in a *min-max* form; usually such theorems involve two parameters one of which is clearly bounded by the other, and the theorem says that these two parameters are equal. For example, Berge observed that in complements of bipartite graphs – that is, graphs whose vertex set can be partitioned into two complete graphs, the chromatic number is always equal to the *clique number*, the largest size of a complete subgraph. While the clique number is an obvious lower bound for the chromatic number, in the paragraph on the 'Probabilistic Method' we explained that in general the chromatic number can be much larger than the clique number. So Berge's observation is an example of a min-max theorem.

Many min-max theorems can be proved through the duality theorem of linear programming; in such cases one of the parameters is determined by a linear maximisation problem and the other parameter can be described by the dual minimisation problem. In fact, Berge's observation is of this form. There are many classes of graphs that allow for a min-max relation between clique number and chromatic number. In the 1960s Berge began to systematically study these classes.

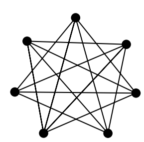


Figure 18: The complement of the 7-cycle. This graph has no K_4-subgraph, yet it is not 3-colourable. So this graph is not perfect.

Recall that an *induced subgraph* of a graph G is obtained by deleting a set of vertices and those edges with at least one endvertex in that set. We say that a graph G is *perfect* if in each of its induced subgraphs, the clique number is equal to the chromatic number, see Figure 18. For example, complements of bipartite graphs are perfect but also bipartite graphs themselves are perfect, as are *chordal* graphs – graphs such that all induced cycles have length three. Put another way, the class of perfect graphs is the class of those graphs such that they and all their induced subgraphs satisfy a min-max relation between clique number and chromatic number.

While min-max theorems tend to have connections with the duality theorem of linear programming, it is a priori not clear that the class of perfect graphs has pleasant algorithmic (or structural) properties. The theory of perfect graphs today answers the following questions. Is there a natural characterisation of the class of perfect graphs in terms of induced subgraphs? In view of the fact that determining the chromatic number of a graph is an NP-hard problem, is there a polynomial algorithm for perfect graphs? Can we recognise a perfect graph in polynomial time?

Recall that the *complement* of a graph G is obtained by flipping the adjacency between vertices; that is, the complement of G is a graph with the same vertex set where two vertices are adjacent if and only if they are not adjacent in G. Proving a conjecture of Berge in 1972, Lovász proved that a graph is perfect if and only if its complement is perfect. Given that the definition of perfect graphs is not symmetric under complementation, this may come as a surprise.

In [38], Grötschel, Lovász and Schrijver constructed a polynomial algorithm to determine the chromatic number of perfect graphs, as follows. In view of the fact that determining the chromatic number or the clique number are NP-hard problems for the class of all graphs, this is just another possibly unexpected aspect of perfect graphs. The key to this algorithm is the *Lovász number*[8] $\theta(G)$ [53]. Lovász carefully designed this geometric graph parameter so that it is essentially 'sandwiched' between the clique number $\omega(G)$ and the chromatic number $\chi(G)$; in formulas $\omega(G) \leq \theta(\overline{G}) \leq \chi(G)$, where $\overline{G}$ is the complement of G. So for perfect graphs, the chromatic number can be computed through the Lovász number of the complement graph. One of the cornerstone results in Combinatorial Optimisation is the fact that the Lovász number can be computed through a semi-definite program, providing a polynomial algorithm to determine the chromatic number of perfect graphs.

Simple examples of graphs that are not perfect are cycles of odd length greater than three – and their complements. In 2006, Chudnovsky, Robertson, Seymour, and Thomas showed that these graphs actually characterise the class of perfect graphs, by showing that a graph is perfect if and only if neither it nor its complement contains an odd cycle of length at least five as an induced subgraph [18]. This is a far-reaching extension of Lovász' theorem that complements of perfect graphs are perfect, and settles a conjecture of Berge. As part of this proof, the authors developed a structural decomposition theory for perfect graphs, which in turn was applied by Chudnovsky, Cornuéjols, Liu, Seymour and Vušković to construct a polynomial recognition algorithm for perfect graphs [16]. It is remarkable that so far it is still

[8] Sometimes the Lovász number is also called the *Lovász theta-function*.

open to find a way how to use these decomposition tools to find a combinatorial algorithm to compute the chromatic number of perfect graphs; see [17] for recent progress in this direction.

Motivated by these successes, in recent years there was a significant interest in understanding graph classes where clique number and chromatic number are not necessarily equal but at least close in a quantified sense. A central conjecture in this area is the following.

Conjecture 4.4. *(Gyárfás–Sumner 1975) For every tree T and natural number r, there is a constant c such that every graph either contains T as an induced subgraph, or contains the complete graph K_r as an induced subgraph, or has chromatic number at most c.*

If true, this conjecture would nicely complement the theorem that there are graphs of large chromatic number such that all bounded-size connected subgraphs are trees, discussed in the paragraph on the 'Probabilistic Method'. In fact this would be a local-global principle for the chromatic number after all. For details, see the survey by Seymour and Scott [83].

Infinite Graph Theory. Recent breakthroughs in this area include the proof of the Erdős-Menger Conjecture by Aharoni and Berger in 2009 [4], Diestel's topological principle which lifts theorems about finite graphs to topological statements about infinite graphs with ends, which climaxed in the solution of Rado's problem by providing cryptomorphic axiomatisations of infinite matroids in terms of independent sets, cases, circuits, hyperplanes and rank [10]; and the subsequent development of a theory of infinite matroids by Bowler, Carmesin, Joó and others. An important conjecture in Infinite Graph Theory is that countable graphs are well-quasi ordered.

While there is no doubt that there is an abundance of natural challenges in the field of Infinite Graph Theory, I see the greatest potential for its development in the next decades in questions that apply Infinite Graph Theory in finite graphs.

Let me illustrate this by an example. Developments in science such as parallel computing and large networks motivate the study of *local separators* – vertex sets that separate graphs locally but not necessarily globally [13]. More precisely, we are interested in small vertex sets whose removal disconnected a subgraph of bounded radius, while outside that subgraph connections may still exist. The key idea in [13] is to define these local separators as those vertex sets that lift to separators of a suitable cover; this way they inherit submodularity properties and many structural decomposition theorems can be extended from separators to local separators, see Figure 19.

There is a subtle, but important, difference to Diestel's topological principle. While Diestel's principle builds exciting problems concerning

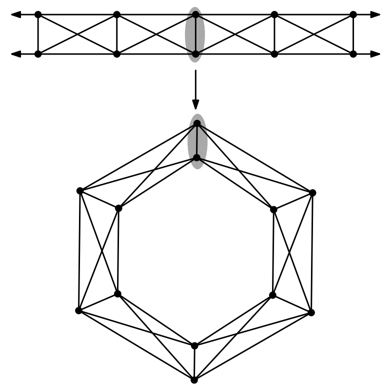


Figure 19: A finite graph covered by an infinite graph. Local separators of finite graphs can be defined through genuine separators of the cover; a local separator and its lift are highlighted in grey.

infinite graphs out of finite graph theory, we go the other way: harness the power of infinite graphs to study local behaviour in finite graphs. There is much to explore in this direction, for example it connects to Open Question 3.4 mentioned above in the paragraph on Graph Limits.

Topological Graph Theory. Above we have seen how the Probabilistic Method can be used to prove the existence of graphs with large chromatic number and large girth, and the 4-Colour Theorem and Hadwiger's Conjecture are concerned with bounding the chromatic number for specific classes of graphs. Lovász developed a method going the other way round, employing algebraic topology to deduce that certain graph classes have high chromatic number. One such class is that of 'Kneser-graphs': given natural numbers n and k, the *Kneser-graph* $K(n, k)$ has as its vertex set all k-subsets of an n-element set, where two vertices are adjacent if their sets are disjoint. M. Kneser conjectured that $K(n, k)$ has chromatic number exactly $n - 2k + 2$ for $n \geq 2k$.

Example 4.5. The Kneser graph $K(5, 2)$ is isomorphic to the Peterson-graph, see Figure 1.

Lovász' approach is based on the following construction. Given a graph G, define the *neighbourhood complex* of G to be the simplicial complex that has the same vertex set as G, whose simplicies are those subsets that are contained in the neighbourhood of a single vertex of G. Lovász [52] proved if the neighbourhood complex is k-connected in the sense of homology theory, then the chromatic number of G is at least $k-1$. An application is the proof of the above mentioned conjecture of Kneser.

Since then a whole research area, usually referred to as *Combinatorial Algebraic Topology*, has grown out of these ideas, for details we refer to book of Kozlov [49], Babson and Kozlov [7] or Ziegler [98]. Aharoni and Berger extended these methods allowing them to study many more graph parameters than just colourings [3], and this work inspired them to propose a matroidal strengthening of Ryser's conjecture [2].

In the 1930s Kuratowski, MacLane, Wagner and Whitney proved various characterisations of the topological property of embeddability of graphs in the plane through algebraic and combinatorial properties; a linear algorithm for testing planarity was discovered by Hopcroft and Tarjan [44] in 1974, and at the end of the 1970s several algorithms had been published that construct plane embeddings in linear time, see [58] for details. The higher dimensional analogue of this problem is to embed 2-dimensional simplicial complex into 3-dimensional space, see Figure 20.

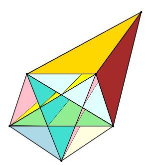


Figure 20: The cone over K_5. The fact that the graph K_5 does not embed in the plane lifts to the non-embeddability of the cone over K_5 in 3-space.

In his survey article on Graph Minor Theory, Lovász asked whether there is an analogue of graph minors in three dimensions. This was answered by Carmesin in 2017, who characterised embeddability of simply connected 2-dimensional simplicial complexes in 3-space in a way analogous to Kuratowski's characterisation of graph planarity, by excluded minors. This characterisation also answered related questions of Pardon and Wagner. A tool in the proof is Perelman's Theorem that any simply connected compact 3-manifold is isomorphic to the 3-sphere.

While Perelman's Theorem is certainly a great advancement in mathematics, the algorithmic aspects of the Poincaré Conjecture remain open: is there a polynomial time algorithm that tests isomorphy with the 3-sphere for triangulated 3-manifolds? While existence of some algorithm was proved by Rubinstein [90][9], Schleimer [81] strengthened this by showing that this problem lies in NP, and Zentner [97]

[9]See for example [81] for details on the history.

showed that this problem lies in co-NP provided the generalised Riemann Hypothesis. An equivalent combinatorial version is the following:

Open Question 4.6. *Is there a polynomial algorithm that tests embeddability in 3-dimensional space for the class of 2-dimensional simplicial complexes whose first homology group over the rationals is trivial?*

Without the assumption on the homology group, de Mesmay, Rieck, Sedgwick and Tancer [60] have shown that such a polynomial algorithm cannot exist. In contrast to this, if we strengthen the assumption requiring that the 2-complex is simply connected, the proof of Carmesin's Kuratowski Theorem gives a quadratic time algorithm [12]. This algorithm relies on Perelman's theorem. It would be desirable to have a combinatorial proof of Perelman's Theorem. Towards this goal Rubinstein, a pioneer in computational topology, asked the following.

Open Question 4.7 (Rubinstein [79])**.** *Is there a robust notion of discrete curvature that allows for a discrete analogue of Ricci flow?*

5 Outlook

In this survey, we made an attempt to explain what graph theory is by sketching some of its main lines of research. By doing so, we were forced to omit many exciting developments in graph theory. For example, an important problem at the interface of group theory, combinatorics and computer science is the *graph isomorphism problem*, which asks whether there is a polynomial time algorithm that decides whether two given graphs are isomorphic. Despite Babai's recent subexponential time algorithm, it is not even known whether this problem is NP-hard. The $P \neq NP$-Conjecture is another problem that we did not explain here although there are many problems in graph theory that are known to be NP-complete, for example deciding whether a graph has a cycle that contains all its vertices [96].

6 László Lovász

In the past century, graph theory emerged as a new area of mathematics. We explained some of its fundamental ideas, its diverse links to other areas of mathematics, as well as today's questions and challenges. Many of these challenges are foundational in nature and are motivated by recent developments in science, intrinsic questions or link to mysteries in other parts of mathematics. Today's graph theorists owe the privilege to work in this exciting area of research to the hard work and deep insights of generations of graph theorists before. In particular László Lovász played a key role through his inspiring ideas, guiding the field. His main contributions include:

- He played a leading role in the development of the Theory of Graphons, which allow us to solve many problems of extremal graph theory through this limit object. A prominent example is the clique density theorem of Reiher, which had been conjectured by Lovász and Simonovits a few decades before.
- The Lovász Local Lemma (LLL) revolutionised Probabilistic Combinatorics, and is a fundamental result in the area of randomised algorithms.
- His topological methods to compute the chromatic number of Kneser graphs were foundational to the field of Combinatorial Algebraic Topology.
- Lovász developed sophisticated tools such as the Lovász Theta function and the Lovász extension of submodular functions. These connect to the ellipsoid method, a central result in optimization, whose applications to combinatorial optimisation were developed by Grötschel, Lovász and Schrijver.
- The Lovász Theta function connects Combinatorial Optimisation with the theory of perfect graphs, to which Lovász also made important contributions.
- He shaped mathematics as President of the International Mathematical Union (2007-2010) and President of the Hungarian Academy of Sciences (2014-2020).
- Beyond solving hard problems, he also wrote comprehensive text books on cutting edge research areas making them accessible to a much broader audience, for example his book 'Large networks and graph limits' or the book on Matching Theory with Plummer.
- Lovász is known for his inspiring questions and conjectures. For example his question on a 3-dimensional graph minor theory is foundational to the development of 3-dimensional Combinatorics. Another puzzle of his is the Erdős-Lovász-Faber conjecture, which has recently been proved by Kang, Kelly, Kühn, Methuku and Osthus.

7 Further reading

Diestel's textbook '*Graph Theory*' is an excellent introduction to the topic, which also treats some of its advanced methods in later chapters.

References

[1] Wilhelm Ackermann. Die widerspruchsfreiheit der allgemeinen mengenlehre. *Mathematische Annalen*, 114(1):305–315, 1937.

[2] Ron Aharoni. Ryser's conjecture for tripartite 3-graphs. *Combinatorica*, 21(1):1–4, 2001.

[3] Ron Aharoni and Eli Berger. The intersection of a matroid and a simplicial complex. *Transactions of the American Mathematical Society*, 358(11):4895–4917, 2006.

[4] Ron Aharoni and Eli Berger. Menger's theorem for infinite graphs. *Invent. math.*, 176:1–62, 2009.

[5] Miklós Ajtai, János Komlós, and Endre Szemerédi. A note on ramsey numbers. *Journal of Combinatorial Theory, Series A*, 29(3):354–360, 1980.

[6] David Aldous and Russell Lyons. Processes on unimodular random networks. *Electronic Journal of Probability*, 12:1454–1508, 2007.

[7] Eric Babson and Dmitry N Kozlov. Proof of the Lovász conjecture. *Annals of Mathematics*, pages 965–1007, 2007.

[8] Tom Bohman and Peter Keevash. The early evolution of the h-free process. *Inventiones mathematicae*, 181(2):291–336, 2010.

[9] Nathan Bowler and Johannes Carmesin. Matroid intersection, base packing and base covering for infinite matroids. *Combinatorica*, 35(2):153–180, 2015.

[10] Henning Bruhn, Reinhard Diestel, Matthias Kriesell, Rudi Pendavingh, and Paul Wollan. Axioms for infinite matroids. *Adv. Math.*, 239:18–46, 2013.

[11] Matija Bucić, Matthew Kwan, Alexey Pokrovskiy, and Benny Sudakov. Halfway to rotas basis conjecture. *International Mathematics Research Notices*, 2020(21):8007–8026, 2020.

[12] Johannes Carmesin. Embedding simply connected 2-complexes in 3-space – V. A refined Kuratowski-type characterisation. Preprint 2017, available at "https://arxiv.org/pdf/1709.04659.pdf".

[13] Johannes Carmesin. Local 2-separators. Preprint, available at `http://web.mat.bham.ac.uk/J.Carmesin/`.

[14] Chandra Chekuri and Julia Chuzhoy. Polynomial bounds for the grid-minor theorem. *J. ACM*, 63(5):Art. 40, 65, 2016.

[15] Maria Chudnovsky. The Erdös–Hajnal conjecture – a survey. *Journal of Graph Theory*, 75(2):178–190, 2014.

[16] Maria Chudnovsky, Gérard Cornuéjols, Xinming Liu, Paul Seymour, and Kristina Vušković. Recognizing Berge graphs. *Combinatorica*, 25:143 – 186, 2005.

[17] Maria Chudnovsky, Aurélie Lagoutte, Paul Seymour, and Sophie Spirkl. Colouring perfect graphs with bounded clique number. *Journal of Combinatorial Theory, Series B*, 122:757–775, 2017.

[18] Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. The strong perfect graph theorem. *Annals of mathematics*, pages 51–229, 2006.

[19] Julia Chuzhoy and Zihan Tan. Towards tight(er) bounds for the excluded grid theorem. *Journal of Combinatorial Theory, Series B*, 146:219–265, 2021.

[20] David Conlon. A new upper bound for diagonal ramsey numbers. *Annals of Mathematics*, pages 941–960, 2009.

[21] David Conlon and William Timothy Gowers. Combinatorial theorems in sparse random sets. *Annals of Mathematics*, pages 367–454, 2016.

[22] Jacob W. Cooper, Daniel Král', and Tasa L. Martins. Finitely forcible graph limits are universal. *Advances in Mathematics*, 340:819–854, 2018.

[23] Reinhard Diestel. *Graph Theory* (5th edition). Springer-Verlag, 2017.
Electronic edition available at:
`http://diestel-graph-theory.com/index.html`.

[24] Reinhard Diestel. Abstract separation systems. *Order*, 35(1):157–170, 2018.

[25] Reinhard Diestel. Tangles in the social sciences. *arXiv preprint arXiv:1907.07341*, 2019.

[26] Reinhard Diestel, Tommy R Jensen, Konstantin Yu Gorbunov, and Carsten Thomassen. Highly connected sets and the excluded grid theorem. *Journal of Combinatorial Theory, Series B*, 75(1):61–73, 1999.

[27] Reinhard Diestel and Sang-il Oum. Tangle-tree duality in abstract separation systems. *Advances in Mathematics*, page 107470, 2020.

[28] Christian Elbracht, Diego Fioravanti, Solveig Klepper, Jakob Kneip, Luca Rendsburg, Maximilian Teegen, and Ulrike von Luxburg. Tangles: from weak to strong clustering. *arXiv preprint arXiv:2006.14444*, 2020.

[29] Paul Erdos and Alfréd Rényi. Asymmetric graphs. *Acta Math. Acad. Sci. Hungar*, 14(295-315):15, 1963.

[30] Hans Freudenthal. Neuaufbau der Endentheorie. *Annals of Mathematics*, 43:261–279, 1942.

[31] Jim Geelen, Bert Gerards, and Geoff Whittle. Solving Rota's conjecture. *Notices Amer. Math. Soc.*, 61(7):736–743, 2014.

[32] Jim Geelen and Peter Nelson. On minor-closed classes of matroids with exponential growth rate. *Advances in Applied Mathematics*, 50(1):142–154, 2013.

[33] Jim Geelen and Peter Nelson. The densest matroids in minor-closed classes with exponential growth rate. *Transactions of the American Mathematical Society*, 369(9):6751–6776, 2017.

[34] Jim Geelen and Sang-il Oum. Circle graph obstructions under pivoting. *Journal of Graph Theory*, 61(1):1–11, 2009.

[35] Archontia C Giannopoulou, Ken-ichi Kawarabayashi, Stephan Kreutzer, and O-joung Kwon. The directed flat wall theorem. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 239–258. SIAM, 2020.

[36] Stefan Glock, Daniela Kühn, Allan Lo, Richard Montgomery, and Deryk Osthus. On the decomposition threshold of a given graph. *Journal of Combinatorial Theory, Series B*, 139:47–127, 2019.

[37] Mikhael Gromov. Endomorphisms of symbolic algebraic varieties. *Journal of the European Mathematical Society*, 1(2):109–197, 1999.

[38] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2. Springer Science & Business Media, 2012.

[39] Andrzej Grzesik, Daniel Král, and László Miklós Lovász. Elusive extremal graphs. *Proceedings of the London Mathematical Society*, 121(6):1685–1736, 2020.

[40] Torbjörn Gustavsson. *Decompositions of large graphs and digraphs with high minimum degree.* PhD thesis, Univ., 1991.

[41] Hamed Hatami, László Lovász, and Balázs Szegedy. Limits of locally–globally convergent graph sequences. *Geometric and Functional Analysis*, 24(1):269–296, 2014.

[42] Zheng-Xu He and Oded Schramm. Hyperbolic and parabolic packings. *Discrete & Computational Geometry*, 14(2):123–149, 1995.

[43] Zheng-Xu He and Oded Schramm. On the convergence of circle packings to the riemann map. *Inventiones mathematicae*, 125(2):285–305, 1996.

[44] John Hopcroft and Robert Tarjan. Efficient planarity testing. *Journal of the ACM (JACM)*, 21(4):549–568, 1974.

[45] Attila Joó. Proof of Nash-Williams' intersection conjecture for countable matroids. *Advances in Mathematics*, 380:107608, 2021.

[46] Ken-ichi Kawarabayashi, Bojan Mohar, and Bruce Reed. A simpler linear time algorithm for embedding graphs into an arbitrary surface and the genus of graphs of bounded tree-width. In *2008 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 771–780. IEEE, 2008.

[47] Peter Keevash. The existence of designs. *arXiv preprint arXiv:1401.3665*, 2014.

[48] Ringi Kim, O-joung Kwon, Sang-il Oum, and Vaidy Sivaraman. Classes of graphs with no long cycle as a vertex-minor are polynomially χ-bounded. *Journal of Combinatorial Theory, Series B*, 140:372–386, 2020.

[49] Dimitry Kozlov. *Combinatorial algebraic topology*, volume 21. Springer Science & Business Media, 2008.

[50] Dávid Kunszenti-Kovács, László Lovász, and Balázs Szegedy. Measures on the square as sparse graph limits. *Journal of Combinatorial Theory, Series B*, 138:1–40, 2019.

[51] Kazimierz Kuratowski. Sur le probleme des courbes gauches en topologie. *Fundamenta mathematicae*, 15(1):271–283, 1930.

[52] László Lovász. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A*, 25(3):319–324, 1978.

[53] László Lovász. On the shannon capacity of a graph. *IEEE Transactions on Information theory*, 25(1):1–7, 1979.

[54] László Lovász. Submodular functions and convexity. In *Mathematical programming the state of the art*, pages 235–257. Springer, 1983.

[55] László Lovász. *Combinatorial problems and exercises*, volume 361. American Mathematical Soc., 2007.

[56] László Lovász. *Large networks and graph limits*, volume 60. American Mathematical Soc., 2012.

[57] László Lovász and Michael D Plummer. *Matching theory*, volume 367. American Mathematical Soc., 2009.

[58] Kurt Mehlhorn and Petra Mutzel. On the embedding phase of the hopcroft and tarjan planarity testing algorithm. *Algorithmica*, 16(2):233–242, 1996.

[59] Karl Menger. Zur allgemeinen Kurventheorie. *Fundamenta Mathematicae*, 10(1):96–115, 1927.

[60] Arnaud de Mesmay, Yoav Rieck, Eric Sedgwick, and Martin Tancer. Embeddability in R3 is NP-hard. *Journal of the ACM (JACM)*, 67(4):1–29, 2020.

[61] Bojan Mohar. A linear time algorithm for embedding graphs in an arbitrary surface. *SIAM Journal on Discrete Mathematics*, 12(1):6–26, 1999.

[62] Robin A Moser. A constructive proof of the Lovász local lemma. In *Proceedings of the forty-first annual ACM symposium on Theory of computing*, pages 343–350, 2009.

[63] Robin A Moser and Gábor Tardos. A constructive proof of the general Lovász local lemma. *Journal of the ACM (JACM)*, 57(2):1–15, 2010.

[64] C St JA Nash-Williams. Edge-disjoint spanning trees of finite graphs. *Journal of the London Mathematical Society*, 1(1):445–450, 1961.

[65] C St JA Nash-Williams. Decomposition of finite graphs into forests. *Journal of the London Mathematical Society*, 1(1):12–12, 1964.

[66] C St JA Nash-Williams. An unsolved problem concerning decomposition of graphs into triangles, combinatorial theory and its applications III. ed. P. Erdös, P. Rényi and VT Sós, 1970.

[67] Seiya Negami. The spherical genus and virtually planar graphs. *Discrete Mathematics*, 70(2):159–168, 1988.

[68] Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity: graphs, structures, and algorithms*. Springer, 2012.

[69] Sang-il Oum. Rank-width and vertex-minors. *J. Combin. Theory, Series B*, 95(1):79–100, 2005.

[70] James Oxley. *Matroid Theory* (2nd edition). Oxford University Press, 2011.

[71] Richard Rado. Universal graphs and universal functions. *Acta Arithmetica*, 9(4):331–340, 1964.

[72] Frank P Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 2(1):264–286, 1930.

[73] Alexander A Razborov. Flag algebras. *The Journal of Symbolic Logic*, 72(4):1239–1282, 2007.

[74] Alexander A Razborov. Flag algebras: an interim report. In *The Mathematics of Paul Erdős II*, pages 207–232. Springer, 2013.

[75] Christian Reiher. The clique density theorem. *Annals of Mathematics*, pages 683–707, 2016.

[76] Neil Robertson, Paul Seymour, and Robin Thomas. Hadwiger's conjecture for K6-free graphs. *Combinatorica*, 13(3):279–361, 1993.

[77] Neil Robertson and Paul D Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 92(2):325–357, 2004.

[78] Burt Rodin and Dennis Sullivan. The convergence of circle packings to the riemann mapping. *Journal of Differential Geometry*, 26(2):349–360, 1987.

[79] J Hyam Rubinstein. Problems around 3-manifolds. In *Workshop on Heegaard Splittings, Geom. Topol. Monogr*, volume 12, pages 285–298, 2007.

[80] Mathias Schacht. Extremal results for random discrete structures. *Annals of Mathematics*, pages 333–365, 2016.

[81] Saul Schleimer. Sphere recognition lies in np. *Low-dimensional and symplectic topology*, 82:183–213, 2011.

[82] Deena R Schmidt and Peter J Thomas. Measuring edge importance: a quantitative analysis of the stochastic shielding approximation for random processes on graphs. *The Journal of Mathematical Neuroscience, available at https://www.researchgate.net/publication/261755434*, 4(1):1–52, 2014.

[83] Alex Scott and Paul Seymour. A survey of χ-boundedness. *Journal of Graph Theory*, 95(3):473–504, 2020.

[84] Jean-Pierre Serre. *Trees*. Springer-Verlag, 1980. (Translated from the French by John Stillwell.).

[85] Paul Seymour. Hadwigers conjecture. In *Open problems in mathematics*, pages 417–437. Springer, 2016.

[86] Alexander Sidorenko. A correlation inequality for bipartite graphs. *Graphs and Combinatorics*, 9(2):201–204, 1993.

[87] Joel Spencer. Ramsey's theorema new lower bound. *Journal of Combinatorial Theory, Series A*, 18(1):108–115, 1975.

[88] John R Stallings. On torsion-free groups with infinitely many ends. *Annals of Mathematics*, pages 312–334, 1968.

[89] Carsten Thomassen. Every planar graph is 5-choosable. *Journal of Combinatorial Theory, Series B*, 62(1):180–181, 1994.

[90] Abigail Thompson. Thin position and the recognition problem for S^3. *Math. Res. Lett.*, 1(5):613–630, 1994.

[91] William Thomas Tutte. *Graph theory as I have known it*. Number 11. Oxford University Press, 1998.

[92] Hein van der Holst. Graphs and obstructions in four dimensions. *Journal of Combinatorial Theory, Series B*, 96(3):388–404, 2006.

[93] Hein Van Der Holst, László Lovász, Alexander Schrijver, et al. The colin de verdiere graph parameter. *Graph Theory and Computational Biology (Balatonlelle, 1996)*, pages 29–85, 1999.

[94] Klaus Wagner. Über eine Eigenschaft der ebenen Komplexe. *Mathematische Annalen*, 114(1):570–590, 1937.

[95] H. Whitney. Non-separable and planar graphs. *Trans. Am. Math. Soc.*, 34:339–362, 1932.

[96] Avi Wigderson. *Mathematics and computation*. Princeton University Press, 2019.

[97] Raphael Zentner. Integer homology 3-spheres admit irreducible representations in sl(2,c). *Duke Math. J.*, 167(9):1643–1712, 2018.

[98] Günter M Ziegler. Generalized Kneser coloring theorems with combinatorial proofs. *Inventiones mathematicae*, 147(3):671–691, 2002.